Unstable quantum systems coupled via continuum and super-radiance

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Abstract

Excited states of a quantum system are unstable and decay into the continuum. The dynamics of the transmission quantum signal through a two-dimensional lattice with open decay channels coupled to the continuum is treated by means of a discretized effective non-Hermitian Hamiltonian. The energies and widths are treated as real and imaginary parts of complex eigenvalues for the effective Hamiltonian. This coupling through the continuum reorganizes the dynamics of the system, as a result the energy widths of the intrinsic states are redistributed and very broad states are formed absorbing a significant part of all the summed energy width. As a result these broad, super-radiant states become highly unstable, with short lifetimes, while the remaining states become trapped and long-lived. This notion of super-radiance was suggested by Dicke, over fifty years ago, for systems pertaining to coherent states in quantum optics. A sharp, sort of phase transition, between weak and strong coupling to the continuum is considered for a two-dimensional open periodic lattice. Due to this continuum coupling a sharp redistribution of energy widths occurs. The weak coupling limit corresponds to isolated sharp resonances, whereas strong coupling corresponds to the collectivization of widths and the formation of the short-lived Dicke state.

Introduction

All excited states in a quantum system are unstable and the notion of a closed physical system is an idealization. Through interactions with the outside world, Heisenberg’s uncertainty principle says that excited states acquire a finite lifetime \(\tau\) and an energy uncertainty (decay width) of \(\Gamma \approx \hbar/\tau\). The standard treatment of quantum mechanics of unstable states introduces their complex energies

\[ \varepsilon = E - i/2 \Gamma \] (1)
States characterized by these complex energies represent the time development of the wave function as \( \sim \exp[-iE - \Gamma/2)t/\hbar] \). This produces an exponential decrease in the probability of the order \( \exp(-\Gamma t/2\hbar) \), as times evolves the state decays. As long as the widths are small compared to the spacing between energy levels, these states can be seen experimentally as resonances in the cross sections of the reactions associated with the decay. It is known that at high energies and high density of the energy levels, the resonances overlap and the pattern becomes more complicated, and statistical methods seem the only method of description [1].

The behavior of a quantum system with \( N \) internal states \( |n\rangle, \ n = 1, \ldots, \ N, \) decaying into \( k \) open channels \( c, \ c = 1, \ldots, k \), can be described by means of an effective non-Hermitian Hamiltonian \( \mathcal{H} \) [2]. The anti-Hermtian part is acquired by the elimination of continuum variables. This so-called effective non-Hermitian Hamiltonian can be written as,

\[
\mathcal{H} = H - \frac{i}{2}W, \tag{2}
\]

where \( H \) is the intrinsic Hermitian Hamiltonian of the system with a discrete energy spectrum and \( W \) originates from the open channels coupled to the continuum. The matrix elements of the anti-Hermitian part \( W \) have a factorizable structure due to the unitarity of the scattering matrix,

\[
W = \sum_c f_n^c f_n'^c \tag{3}
\]

where \( f_n^c \) and \( f_n'^c \) are the decay amplitudes, coupling the state \( |n\rangle \) to the open channel \( c \).

The Hamiltonian \( \mathcal{H} \) acts in the intrinsic \( N \)-dimensional space, and is obtained by a projection of the of the full Schrödinger equation onto a subspace of the intrinsic states. One of the practical advantages of using an effective Hamiltonian acting in the intrinsic subspaces only is that standard quantum mechanics matrix methods can be applied, as those developed for the discrete intrinsic spectrum with no decay amplitudes. For further discussion into the properties of the effective Hamiltonian, the reader is directed to references [2,3].
**Dicke coherent States**

The idea of Dicke was confirmed in the seventies by observation of super-radiance in the transmission of a laser pulse through specific media. Much later was it realized that the mechanism of super-radiance arises in many other areas of physical phenomena in atomic, nuclear and particle physics. In this case we applied the concept to quantum systems, where internal states are unstable and can undergo irreversible decay into the continuum. It is important that this consideration does not violate basic quantum laws, such as the conserved hermicity of the intrinsic Hamiltonian and specially the unitarity of the scattering matrix.

Two limiting cases correspond to weak and strong coupling to the continuum of the intrinsic states. In the weak coupling limit, the spacings $D$ between the discrete energy levels are large compared with the decay widths $\langle \Gamma \rangle$. If this is the case, the anti-Hermitian part $W$ can be treated as a perturbation of the intrinsic Hamiltonian $H$. The main effect of its diagonal elements is the instability of the intrinsic states. If the physical coupling parameter

$$\mu = \frac{\langle \Gamma \rangle}{D}$$

is small, that is $\mu \ll 1$, the off-diagonal terms $W$ are then of minor importance and the energy spectrum consists of well isolated narrow resonances. It should be noted that $\langle \Gamma \rangle$ and $D$ in equation (4) are an average width and spacing, respectively. In the opposite case of strong coupling with the continuum, $\mu \gg 1$, one would expect naively that the states would overlap. For this case, $W$ is the main part of the Hamiltonian and $H$ can be treated as a perturbation. Due to its factorized form, the operator $W$ has only one non-zero eigenvalue for only one channel. Because the unitarity of the scattering matrix is guaranteed, the only non-zero eigenvalue $\Gamma_D$ accumulates the total width of all the summed intrinsic states, and the trace of $W$ is conserved

$$\Gamma_D = Tr \, W = \sum_n f_n^2 \equiv w$$

The remaining trapped states, corresponding to the degenerate zero eigenvalues of $W$, become stable and the whole decay strength is accumulated by the coherent super-radiant state [4]. It
should be mentioned that the stable trapped states acquire a non-zero decay probability due to the presence of the Hermitian interaction $H$ [1]. This collectivization due to the coupling through the continuum is quite similar in its mechanism to the formation of collective giant modes in nuclei [5] or plasmons in electron systems [6]. But in these cases the collectivization occurs along the imaginary axis in the complex energy plane.

**Open Periodic Two Dimensional Array**

A periodic array of $N \times M$ identical potential wells ($n = 1, 2, ..., N$ and $m = 1, 2, ..., M$), in two dimensions is considered, so that the intrinsic states are now labeled $|nm\rangle$. This can be thought of a simple model for particle transport. These are coupled by an interaction $\nu$ of the particle to hop or tunnel, vertically or horizontally to the nearest neighbor well. Each state is coupled to the continuum by the escape probability amplitude $A_{nm}$ of the particle to decay, thus escaping the potential well from the site $nm$. If each isolated well supports the energy level $\varepsilon$, the effective Hamiltonian (2) can be written as a matrix in the site representation,

$$
\mathcal{H}_{nm,n'm'} = \varepsilon \delta_{n',n} \delta_{m',m} + \nu \left( \delta_{n+1,m,n'm'} + \delta_{n-1,m,n'm'} + \delta_{nm+1,n'm'} + \delta_{nm-1,n'm'} \right) - \frac{i}{2} A_{nm} A_{n'm'} \quad (6)
$$

Careful revision of the subscripts on $\mathcal{H}$ determine its physical significance; reading from left to right, a particle introduced into the lattice in the site $n,m$ would have energy $\varepsilon$, for the particle at this site there exist a probability $\nu$ to tunnel into the four nearest neighbor sites $n \pm 1, m$ or $n, m \pm 1$. The last term introduces a decay probability to open channels for each site. Comparison with equation (2) should make obvious the correspondence of each term.

The intrinsic internal problem (no decay), is commonly used in solid state physics to model slowly moving heavy atomic cores and conducting electrons exposed to the Coulomb potential of the stationary cores. The tunneling coupling of the localized states $|nm\rangle$, produces the Bloch $|q_nq_m\rangle$ waves with quantized wave vectors $q_n = 1, 2, ..., N$ and $q_m = 1, 2, ..., M$. Then the intrinsic Hamiltonian
\[ H = \varepsilon \delta_{n,m} + v(\delta_{n+1,m} + \delta_{n-1,m} + \delta_{nm} + \delta_{n-1,m+1} + \delta_{n+1,m-1}), \]  

(7)  

can be formulated in the localized states space. Schrödinger’s equation,

\[ H|nm\rangle = E|nm\rangle \]  

(8)

gives the energy eigenvalues of the stable system. Expanding the localized states in the Bloch wave basis such that,

\[ |nm\rangle = \sum_{q,q_n} C_{q,q_n} |q_n q_m\rangle \]  

(9)

Applying the new basis to equation (8) and multiplying by an orthogonal vector \( \langle q_n' q_m' | q_n q_m \rangle \), such that \( \langle q_n' q_m' | q_n q_m \rangle = \delta_{q_n q_n' q_m q_m} \), it is obtained

\[ \sum_{q,q_n} C_{q,q_n} \langle q_n' q_m' | H | q_n q_m \rangle = E \sum_{q,q_n} C_{q,q_n} \langle q_n' q_m' | q_n q_m \rangle \]  

(10)

Using the orthogonality property of the vectors and proposing that,

\[ C_{q,q_n} = e^{i\varphi_n} e^{i\varphi_m}, \quad \varphi_n = \frac{\pi q_n}{N+1}, \quad q_n = 1, \ldots, N \text{ and } \varphi_m = \frac{\pi q_m}{N+1}, \quad q_m = 1, \ldots, M \]  

(11)

It is obtained that the \( N \times M \)-fold degenerate level \( \varepsilon \) is split into a band, of standing waves, of width \( 8v \) with energies

\[ E_{q,q_n} = \varepsilon + 2v(\cos \varphi_n + \cos \varphi_m) \]  

(12)

This result is that typically found in solid state physics of the band structure for the spectrum eigenenergies [7]. For future calculations with equation (12), we put \( \varepsilon = 0 \), centering the energy at the origin. In order to find the quasi-stationary states of the system, we take into account the decay amplitudes in the Hamiltonian (6). For simplicity of notation we take \( q_n q_m = q \) and \( q_n' q_m' = q' \). As long as no confusion shall arise, all sums over \( q \) and \( q' \) will be double sums over both subscripts. The decay part of the Hamiltonian defined as \( W_{q,q_n q',q_m'} = f_{q,q_n} f_{q',q_m'} \), such that the Bloch waves are coupled to the continuum by the amplitudes
\[ f_{q,nm} = \sum_{nm} A_{nm} \langle nm | q_n q_m \rangle, \]  
\[ \text{(13)} \]

with \( A_{nm} \) being the decay probability from the site \( nm \).

The normalized wave function on the site representation of the stationary Bloch waves, with zero boundary conditions for the lattice sides, \( n = m = 0, \ n = N + 1, \ m = M + 1 \) is,

\[ \langle nm | q_n q_m \rangle = \frac{2}{\sqrt{(N+1)(M+1)}} \sin n \varphi_{q_n} \sin m \varphi_{q_m}, \]  
\[ \text{(14)} \]

where \( \varphi_{q_n} \) and \( \varphi_{q_m} \) are as defined by equation (11). Back to the problem at hand, expanding the effective Hamiltonian eigenfunction \( |\Psi\rangle \) in the Bloch wave basis \( |q'_n q'_m\rangle = |q\rangle \). Then just as in the no decay case

\[ |\Psi\rangle = \sum_{q'} C_{q'} |q\rangle \]  
\[ \text{(15)} \]

In methods very similar to those taken to calculate the stationary states eigenenergies (12), the secular equation for the complex energies \( \mathcal{E} \) of the effective Hamiltonian, for the two dimensional lattice is,

\[ 1 = i/2 \sum_q \frac{f_q^2}{E - E_q} \]  
\[ \text{(16)} \]

The energies \( \mathcal{E} \) of the system are the complex roots of equation (16).

Accordingly this equation, using (1), can be written as a pair of coupled real equations

\[ \sum_q f_q^2 \frac{E - E_q}{(E - E_q)^2 + \Gamma^2/4} = 0, \]  
\[ \frac{\Gamma}{4} \sum_q \frac{f_q^2}{(E - E_q)^2 + \Gamma^2/4} = 1 \]  
\[ \text{(17)} \]
In the strong continuum coupling regime, for the broad state $E_D$ the resonance is centered at an average energy of the intrinsic states weighted by the total of the original widths,

$$E_D = \frac{\sum_q E_q f_q^2}{w} \equiv \langle \epsilon \rangle$$  \hspace{1cm} (18)

The width of the broad state differs from the total width $Tr W = w$ by an amount related to the mean square spread of the original energies, weighted again with the original widths

$$\Gamma_D = w - \frac{4}{w} \left[ \langle \epsilon^2 \rangle - \langle \epsilon \rangle^2 \right].$$  \hspace{1cm} (19)

Since $W$ is conserved, the difference $w - \Gamma_D$ accounts for the energy widths of the remaining $(N \times M - 1)$ trapped states. The coupling of the intrinsic states through continuum introduces a new energy interpretation for the system that can be visualized with the associated time intervals. The shortest time $\tau \sim \hbar/\Gamma_D$, characterizes a fast direct reaction, while the small width of the trapped states determines the lifetime of the equilibrated system.

**Uniform Decay**

One particular case for the system is that where the decay probabilities are the same for all sites, $A_{nm} = A = \text{const}$. For this case, $f_q$ is the double sum,

$$f_q = \frac{2A}{\sqrt{(N+1)(M+1)}} \sum_{nm} \sin n\varphi_{q_n} \sin m\varphi_{q_m}$$  \hspace{1cm} (20)

For all states with even $q_n$ or $q_m$ the decay probability vanishes, this is characteristic of a Fourier sine expansion of constant amplitude. These Bloch waves become fully trapped and, because of equation (5), the full decay strength is conserved.

The simplest non-trivial case is a system of nine wells $(N = M = 3)$, where the energies of the Bloch waves are those given by equation (12), taking $\epsilon = 0$, as stated earlier. The only non-
vanishing intrinsic energies are \( E_{11} = 2\sqrt{2}v \) and \( E_{33} = -2\sqrt{2}v \). For this case, the decay amplitudes of all the states are

\[
\begin{align*}
 f_{11} &= \frac{A}{2} \left( \sqrt{2} + 1 \right)^2 \\
 f_{12} &= 0 \\
 f_{13} &= \frac{A}{2} \\
 f_{21} &= 0 \\
 f_{22} &= 0 \\
 f_{23} &= 0 \\
 f_{31} &= \frac{A}{2} \\
 f_{32} &= 0 \\
 f_{33} &= \frac{A}{2} \left( \sqrt{2} - 1 \right)^2
\end{align*}
\]  

(21)

Five out of the nine states are decoupled from the continuum and we are left with an effective four state problem, all of the states of even \( q \) or \( m \) became trapped. The symmetry between the states \( f_{13} \) and \( f_{31} \), and the fact that both \( E_{13} \) and \( E_{31} \) are both vanishing energies produces equal denominators when expanding (16), thus reducing the degree of the energy polynomial from a fourth degree to a third degree polynomial. For this case, equation (16) allows obvious exact solutions for the complex energies \( \mathcal{E}/v = \mathcal{E}_v \) as the roots of the cubic equation,

\[
\mathcal{E}^3 + \frac{9k}{2} i \mathcal{E}^2 + \mathcal{E}_v (12ik - 8) - 2ik = 0
\]  

(22)

In equation (22) the parameter \( v \) is used as a scaling factor for the complex energies and does not change the physical aspects of the problem. The coupling parameter \( k = A^2/v \) measures the degree of coupling of the system to the continuum. If the system is in the weak continuum coupling regime, taking the limit of \( k \ll 1 \) the real energies, along \( E = 0 \), are closed to the unperturbed intrinsic energies,

\[
E_{\pm} = \pm 2\sqrt{2} \left[ 1 - \left( \frac{k}{16} \right)^2 \right]
\]  

(23)

The widths of these states are equal to the non-vanishing intrinsic energies plus a small correction of the order of \( k^2 \). This perturbation leads to level attraction, contrary to the usual “mixing” of the energies produced by perturbations to a system. Such attraction is a signature towards the trend of width collectivization characteristic of unstable systems.
In the strong coupling limit, $k \gg 1$, the results for the energies are as given by equations (18) and (19). The broad state is located at $E_D \approx (8/3)v$, and its width having collected the lion’s share of the total width is close to $w$.

$$\Gamma_D \approx w\left[1 - 4/(81k^2)\right].$$  \hfill (24)

The remaining trapped states have a miniscule total width of $\Gamma_T \approx 4/(81k^2)$.

Figure 1 illustrates the behavior of the complex energies, equation (22), of the case $N = M = 3$ as a function of the parameter $k = 1, 2, \ldots, 10$. It is seen that as $k$ increases (the coupling becomes stronger) two of the three states become trapped, while the state $\mathcal{E}_2$ becomes super radiant, as width collectivization occurs. This is clearly seen because $\Gamma_2$ grows unbounded as the coupling parameter becomes larger. As the continuum coupling constant $k$ increases, the state $\mathcal{E}_1$ acquires a small width (perturbation region) before becoming trapped again. The state $\mathcal{E}_3$ is tightly bounded to the system, as its width remains constantly close to zero.

**Figure 1.** Complex energies for the case $N = M = 3$ and uniform decay. (a) The behavior of the real parts of the roots for the complex energy $\mathcal{E}_v$ as a function of $k$. (b) The imaginary parts of the roots of $\mathcal{E}_v$ as a function of $k$. The subscripts 1, 2 and 3 for $\Gamma$ and $E$ refer to each of the three roots of equation (22).
Decay From the Corners

Let the periodic potential chain contain the same characteristics as in the preceding case, but only open at the corner sites, so that only these sites have access to the continuum. More formally the open channels are: \( n = m = 1; n = N, m = 1; n = 1, m = M \) and \( n = N, m = M \). The decay probabilities of the four corner sites are \( A_{11}^2 = \gamma_{11}, A_{N1}^2 = \gamma_{N1}, A_{1M}^2 = \gamma_{1M} \) and \( A_{NM}^2 = \gamma_{NM} \). Then, the anti-Hermitian matrix elements of the effective Hamiltonian in the Bloch wave representation, \( \langle nm|q_n q_m\rangle \), are given by equation (3). It can be shown that in the weak coupling limit each Bloch wave acquires a width

\[
\Gamma_{q_n q_m} = W_{q_n q_m} = \frac{4}{(N+1)(M+1)}(\gamma_{11} + \gamma_{N1} + \gamma_{1M} + \gamma_{NM})\sin^2\varphi_{q_n}\sin^2\varphi_{q_m}
\]  

(25)

This result can be more clearly written as

\[
\Gamma_{q_n q_m} = \gamma_{11}\langle 11|q_n q_m\rangle^2 + \gamma_{N1}\langle N1|q_n q_m\rangle^2 + \gamma_{1M}\langle 1M|q_n q_m\rangle^2 + \gamma_{NM}\langle NM|q_n q_m\rangle^2,
\]  

(26)

which forms a more intuitive physical picture; the continuum coupling singles out those quasistationary states which are connected to the appropriate decay channels. This the same argument used in the derivation of the Porter-Thomas distribution for the width distribution of compound states [8].

In the strong coupling regime the corner widths are large compared to the tunneling coupling \( \nu \), and the above arguments become invalid. The coupling changes the intrinsic structure introducing important corrections to the wave functions of the quasistationary states. Sokolov and Zelevinsky [2] treated a similar problem for a one-dimensional periodic array of \( N \) wells open only at the corner sites. The problem reproduces the same physical effects for the two-dimensional analog, but the mathematics and language behind it are far less cumbersome. For this case if the edge widths \( \gamma_{L,R} \) (from the open corners) are large compared to the coupling \( \nu \), two collective states with large imaginary energies, \( \Gamma_{L,R} = \gamma_{L,R} \), is obtained. Then according to the open number of channels, two short-lived Dicke states located in the middle of the spectrum superimposed in the middle of \((N-2)\) long-lived states are obtained. The edge states accumulate the entire width as irreversible decay occurs much faster than tunneling to the next well. In this
sort of phase transition the short-lived surface-localized states segregate from the band of the delocalized Bloch waves.

Conclusions

Under the influence of continuum coupling systems undergo a transition, where the intrinsic states undergo a sudden restructuring. The system is rearranged in such a way that the decay probabilities single out the appropriate quasistationary states, in such a way that these match the decay channels in an optimal way. At some point, determined by the coupling parameter, the rearrangement leads to the separation of a state, identified with a fast process that accumulates a large fraction of the total combined width. The rest of the states become trapped and long lived, although some with a non-zero decay probability. The considered model of a two dimensional array of regular potentials coupled to the continuum presented results similar in character as the Dicke super-radiance phenomenon. The strong coupling with the continuum leads to a sharp redistribution of widths creating a short-lived state. Similarly if there exist $c$ open channels with similar decay probabilities, $c$ broad states are forms by absorbing practically the whole summed width.

One can certainly imagine systems such as the ones treated here having possible applications. One particularly interesting example is that of information storage for quantum computing. Having long lived states that survive interactions with its environment avoids the problem of decoherence. The environment will tend to couple to and suppress interference between a preferred set of states, thus preserving quantum information. These preferred states can be characterized in terms of their ‘robustness’ or ‘stability’ with respect to the interaction with the environment. Roughly speaking, while the system gets entangled with the environment, the states between which interference is suppressed are the ones that get least entangled with the environment themselves under further interaction [9], such as a measurement of the state to obtain its encoded information.
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References


